

Slowly changing potential problems in Quantum Mechanics: Adiabatic Theorems, Ergodic Theorems, and Scattering

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Abstract

We employ the recently developed multi-time scale averaging method to study the large time behavior of slowly changing (in time) Hamiltonians. We treat some known cases in a new way, such as the Zener problem, and we give another proof of the Adiabatic Theorem in the gapless case. We prove a new Uniform Ergodic Theorem for slowly changing unitary operators. This theorem is then used to derive the adiabatic theorem, do the scattering theory for such Hamiltonians, and prove some classical propagation estimates and Asymptotic Completeness.

1 Introduction

The dynamics of a system perturbed by a time-dependent potential is difficult to analyze in general terms. This problem shows up in many fields as adiabatic processes in Quantum Mechanics, as effective theories in scattering and nonlinear dynamics, in noisy and/or periodically perturbed systems, and more.

A “special” case of fundamental importance is the possibility of identifying a small parameter that controls the rate of change of the system/perturbation. The most celebrated example is the *adiabatic theorem* in QM e.g., [12, 14], where one assumes that the perturbing potential, W , changes with time as

$$W(x, \epsilon t) \tag{1.1}$$

where ϵ is assumed small, *and* the time dependence vanishes after some finite time, that is $W(x, \epsilon t) = W(x, \infty)$ for $\forall t > t_\infty$ typically t_∞ is of order $1/\epsilon$. However, many other examples exist where the slow change never really stops, or continues for a time much larger than $1/\epsilon$. In this case, the standard adiabatic theorem fails [5, 6]

In this work we will study three different examples. The Landau-Zener(LZ) type Hamiltonian, the gapless adiabatic theorem in Quantum Mechanics and scattering theory for some slowly changing potentials.

The LZ model is discussed in section 3. We study in detail the behavior in time for small, medium and large times, using our new multi-scale time averaging method [1]. The method we use is of general nature, can be applied in a similar way to the general form of such Hamiltonian.

In particular we show the time scales on which the LZ Hamiltonian has a nontrivial action, with error estimates. Then, in section 4 we consider the adiabatic theorem in Quantum mechanics. First we prove a uniform adiabatic ergodic theorem. That is we show that for Hamiltonians which change slowly in time, the RAGE theorem holds. See theorem 4.

Then it is applied to give a new proof of the gapless adiabatic theorem.

In section 5 we consider the scattering problem on a short-range slowly changing(i.e adiabatic) potentials.

We prove some basic propagation estimates and then for a class of potentials, we show that the limits defining the scattering matrix are uniform in $\epsilon \rightarrow 0$ provided the potential perturbation $V(x, \epsilon t) = V(x, \infty)$ for $t \geq c \exp(\epsilon^{-1/4})$

An important example is the Landau-Zener type Hamiltonians, the simplest of which is the 2×2 case given by

$$H(t) = \begin{pmatrix} \epsilon t & B \\ B & -\epsilon t \end{pmatrix} \text{ acting on } \mathbf{C}^2. \quad (1.2)$$

In this work, we study the problem of slowly changing potentials, with or without the assumption of finite “life-time”. We consider in detail the LZ model: our aim is to prove that it is asymptotically stable/complete, in the sense that the asymptotic Hamiltonian is an explicit, time-independent operator. Moreover, we find the *interaction time* [17, 18, 13] that is, the time after which the dynamics is time-independent up to *explicitly given* small correction.

We also analyze the short time behavior of the system. It turns out to be oscillatory. We then analyze other problems that can be treated by the same approach: scattering theory and the adiabatic case. For this, we prove a new Uniform Ergodic Theorem.

The method we employ is a multiscale time averaging technique, recently developed by us [1]. In this approach, we study by time-averaging, the dynamics given by

$$i \frac{\partial \psi}{\partial t} = \beta A(t) \psi \quad (1.3)$$

where $|\beta| \ll 1$, $A(t)$ is a family of self-adjoint operators acting in a Hilbert space \mathcal{H} with initial condition $\psi(t=0) = \psi \in \mathcal{H}$.

We show that by averaging over a time interval of order $\frac{1}{\sqrt{\beta}}$, followed by a normal form transformation, the above problem is replaced by a new, piecewise time-independent problem generated by the averaged Hamiltonian, and a correction generated by a system

as above, but with $A(t) \rightarrow \tilde{A}(t)$ and $\beta \rightarrow C\beta^{\frac{3}{2}}$. This process can be redone repeatedly, to all orders in β .

2 Slowly changing potentials and time averaging

Consider the problem

$$\begin{aligned} i \frac{\partial \psi}{\partial t} &= (-\Delta + W(\varepsilon t, x)) \psi \\ \psi(t=0) &= \psi_0 \in L_2(\mathbb{R}^n), \end{aligned} \quad (2.1)$$

where $W(\varepsilon t, x)$ is a smooth bounded function such that for some $\sigma > \sigma_0 > 1$, $0 < \varepsilon \ll 1$

$$\sup_t \|\langle x \rangle^\sigma W(\varepsilon t, x)\|_{L^\infty} < C_0 < \infty \quad (2.2)$$

Here

$$\langle x \rangle^2 = 1 + |x|^2. \quad (2.3)$$

We are interested in the behavior of the solutions of such an equation, for $t > 1$. We assume

$$\sup_\tau \left\| \langle x \rangle^\sigma \frac{\partial W(\tau, x)}{\partial \tau} \right\|_{L^\infty} < C_1 < \infty. \quad (2.4)$$

The solution exists for all times, in L^2 . See Reed-Simon [15]. Traditionally, this problem is treated by the adiabatic theory. Here, we will show how to treat this and similar problems by multiscale time averaging techniques. We will use the multiscale time-averaging of [1]. Let $T_0 = 1/\sqrt{\varepsilon}$.

Define

$$\overline{W}(x) = \frac{1}{T_0} \int_0^{T_0} W(\varepsilon t, x) dt = \frac{1}{\varepsilon T_0} \int_0^{\varepsilon T_0} W(\tau, x) d\tau \quad (2.5)$$

Then

$$\begin{aligned} W(\varepsilon t, x) - \overline{W}(x) &= \frac{-1}{\varepsilon T_0} \int_0^{\varepsilon T_0} [W(\tau, x) - W(\varepsilon t, x)] d\tau \\ &= \frac{-1}{\varepsilon T_0} \int_0^{\varepsilon T_0} W'(y(\tau), x)(\tau - \varepsilon t) d\tau \end{aligned} \quad (2.6)$$

$$y = y(\tau),$$

where we used that $W(\tau, x) - W(t, x) = \frac{\partial W}{\partial y'}(y', x) \Big|_{y'=y(\tau)} (\tau - t)$ for some $y(\tau) \in [\tau, t]$. For $t \leq T_0$, one has $|\tau - \varepsilon t| \leq \varepsilon T_0$, and therefore, we have:

Lemma 1.

$$\sup_{t \leq T_0} \sup_x |W(\varepsilon t, x) - \overline{W}(x)| \langle x \rangle^\sigma \leq C_1 \varepsilon T_0 = C_1 \sqrt{\varepsilon}. \quad (2.7)$$

Hence, if we let (see [1])

$$\overline{W}_j(x) = \frac{1}{T_0} \int_{jT_0}^{(j+1)T_0} W(\varepsilon t, x) \, dt, \quad (2.8)$$

then for $jT_0 \leq t \leq (j+1)T_0$

$$H(\varepsilon t) - (-\Delta + \overline{W}_j(x)) = \mathcal{O}(\langle x \rangle^{-\sigma}) \sqrt{\varepsilon}. \quad (2.9)$$

Let

$$\tilde{A}(t) = \left(H(\varepsilon t) - H_0^{(g)}(\varepsilon t) \right) \varepsilon^{-1/2} \quad (2.10)$$

where

$$H_0^{(g)}(\varepsilon t) = -\Delta + \overline{W}_j(x) \text{ for } jT_0 \leq t \leq (j+1)T_0. \quad (2.11)$$

Then, by (2.9), $\|\langle x \rangle^\sigma \tilde{A}(t)\|_{L^2 \cap L^\infty} \equiv \left\| \langle x \rangle^\sigma \tilde{A}(t) \right\|_{L^2} + \left\| \langle x \rangle^\sigma \tilde{A}(t) \right\|_{L^\infty} = C(C_0 + C_1) \leq O(1)$, where C is a universal constant independent of the Hamiltonian. Therefore, we can rewrite (2.1) in the form

$$i \frac{\partial \psi}{\partial t} = \left[\varepsilon^{1/2} \tilde{A}(t) + H_0^{(g)}(\varepsilon t) \right] \psi(t). \quad (2.12)$$

This implies that

$$\psi(t) = V(t) \tilde{\psi}(t) \quad (2.13a)$$

with time ordering (see Eq. 2.3 [1])

$$V(t) = \mathcal{T} e^{-i \int_0^t H_0^{(g)}(\varepsilon s) \, ds} \quad (2.13b)$$

and

$$\left(\varepsilon^{1/2} \tilde{A}(t) + H_0^{(g)} \right) \psi = i \frac{\partial \psi}{\partial t} = H_0^{(g)}(\varepsilon t) \psi(t) + V(t) i \frac{\partial \tilde{\psi}}{\partial t}. \quad (2.14)$$

The second equality is found by direct differentiation of (2.13a) Therefore

$$V(t) i \frac{\partial \tilde{\psi}}{\partial t} = \varepsilon^{1/2} \tilde{A}(t) V(t) \tilde{\psi}(t) \quad (2.15)$$

or

$$i \frac{\partial \tilde{\psi}}{\partial t} = \varepsilon^{1/2} V(t)^{-1} \tilde{A}(t) V(t) \tilde{\psi}(t) \equiv \beta A(t) \tilde{\psi}(t). \quad (2.16)$$

We have used 2.13b which implies

$$\psi(t) = \mathcal{T} e^{-i \int_0^t H^{(g)}(\varepsilon s) \, ds} \tilde{\psi}(t) \quad (2.17)$$

and then by

$$V(t)^{-1} \tilde{A}(t) V(t) \equiv A(t); \quad \beta \equiv \sqrt{\varepsilon}. \quad (2.18)$$

We now apply the multiscale-averaging of [1] to Equation (2.16) where $A(t)$ plays the role of the Hamiltonian. By Theorem 3 there:

$$\tilde{\psi}(t) \equiv U_0(t) U_1(t) \tilde{U}_2^{-1} \phi_2^U(t) \quad (2.19a)$$

$$i \frac{\partial}{\partial t} \phi_2^U(t) = \beta^{3/2} A_{NF}(t) \phi_2^U(t) \quad (2.19b)$$

$$\|A_{NF}(t)\| = \mathcal{O}(1). \quad (2.19c)$$

Where A_{NF} is obtained by a normal form transformation of the potential averaging, A_{NF} is denoted by \tilde{A} and it is given by (2.18) and $\phi_2^U = \psi(0)$

Here $U_0(t)$ is generated by

$$i \partial_t U_0(t) = \beta \overline{A}_0^g(t) U_0(t) \quad (2.20)$$

where

$$\overline{A}_0^g(t) \equiv \overline{A}_0^{(n)}, \quad \text{for } nT_0 \leq t < (n+1)T_0 \quad (2.21)$$

and

$$\overline{A}_0^{(n)} \equiv \frac{1}{T_0} \int_{nT_0}^{(n+1)T_0} A(s) \, ds, \quad T_0 = \beta^{-1/2}. \quad (2.22)$$

$U_1(t)$ is generated by $\beta \overline{A}_1^g(t)$ with

$$\overline{A}_1^{(n)} \equiv \frac{1}{T_0} \int_{nT_0}^{(n+1)T_0} U_0^{-1}(t) [A(t) - \overline{A}_0^g(t)] U_0(t) \, dt \quad (2.23)$$

$$U_2 \equiv 1 + i\beta U_1^{-1}(t) \left(\int_0^t (A(t') - \overline{A}_1^g(t')) \, dt' \right) U_1(t) = \mathcal{O}(\beta t); |t| \leq \frac{1}{\sqrt{\varepsilon}}. \quad (2.24)$$

Obtained by a model form transformation.

Next to the leading order we get approximating A_{NF} by 1 and it implies that ϕ_2^U is a constant therefore using (2.24) and taking U_2 in the leading order implies

$$\psi(t) = V(t) U_0(t) [U_1(t) \psi_0 - U_1(t) \left(i\beta \int_0^t (A(t') - \overline{A}_1^g(t')) \, dt' \right) \psi(0) + \mathcal{O}(\beta^2)] + \mathcal{O}(\beta^{3/2}t) \quad (2.25)$$

substituting $V(t)$, U_0 and U_1 implies

$$\psi(t) = \mathcal{T} e^{-i \int_0^t H_0^{(g)}(t') dt'} \mathcal{T} e^{-i\beta \int_0^t \overline{A}_0^g(t') dt'} \mathcal{T} e^{-i\beta \int_0^t \overline{A}_1^g(t') dt'} (U_2^{-1} \psi(0)) + \mathcal{O}(\beta^{3/2}t) \quad (2.26)$$

In order to simplify this, we use the following Lemma

Lemma 2.2

Let A and B be self adjoint. Then

$$O \equiv \mathcal{T}e^{i \int_0^t A(s)ds} \mathcal{T}e^{i \int_0^t B(s)ds} = \mathcal{T}e^{+i \int_0^t C(t')dt'} \quad (2.27)$$

where

$$C(t) = A(t) + \mathcal{T}e^{i \int_0^t A(s)ds} B(t) \mathcal{T}e^{-i \int_0^t A(s)ds} \quad (2.28)$$

Proof: Differentiation of O with respect to t yields

$$\frac{\partial}{\partial t} O = A(t) \mathcal{T}e^{i \int_0^t A(s)ds} \mathcal{T}e^{i \int_0^t B(s)ds} + \mathcal{T}e^{i \int_0^t A(s)ds} B(t) \mathcal{T}e^{i \int_0^t B(s)ds} = \left[A(t) + \mathcal{T}e^{i \int_0^t A(s)ds} B(t) \mathcal{T}e^{-i \int_0^t A(s)ds} \right] O \quad (2.29)$$

identifying the term in the square brackets with $C(t)$ completes the proof of the Lemma. \square

3 Landau Zener Majorana Example

The Landau Zener problem is defined by the Schrödinger equation [2, 3, 14, 13, 17, 18, 16, 12, 5, 6].

$$i \frac{\partial}{\partial t} \psi = H(\varepsilon t) \psi \quad (3.1)$$

where ψ is chosen to be a spinor and the Hamiltonian is

$$H(\varepsilon t) = \begin{bmatrix} \varepsilon t & B \\ B & -\varepsilon t \end{bmatrix} = \varepsilon t \sigma_z + B \sigma_x \quad (3.2)$$

where σ_x, σ_y and σ_z are the Pauli matrices. This is a two state system (or its finite dimensional generalizations) The adiabaticity parameter is ϵ . Therefore in the present work it will be assumed to be small. The time-averaged Hamiltonian in the interval $nT_0 \leq t \leq (n+1)T_0$ is given by

$$\overline{H}_n = \begin{bmatrix} \varepsilon T_0 \left(n + \frac{1}{2}\right) & B \\ B & -\varepsilon T_0 \left(n + \frac{1}{2}\right) \end{bmatrix} \quad (3.3)$$

and in this interval $H_0^{(g)}$ of (2.11) takes this value.

The propagator of (2.13a) is, for $nT_0 < t < (n+1)T_0$, given by

$$V(t) = e^{-i\overline{H}_{N_t}(t)} f_{n-1} \cdots f_1 f_0 \quad (3.4)$$

where

$$f_n \equiv e^{-i\overline{H}_n T_0}. \quad (3.5)$$

The effective Hamiltonian of Equation (2.18), is

$$A(t) = V(t)^{-1} \left[H(\varepsilon t) - H_0^{(g)}(t) \right] V(t). \quad (3.6)$$

In the interval $N_t T_0 < t < (N_t + 1)T_0$ the term in the square brackets is

$$H(\varepsilon t) - \overline{H}_{N_t} = \begin{bmatrix} \varepsilon t & B \\ B & -\varepsilon t \end{bmatrix} - \begin{pmatrix} \varepsilon T_0 (N_t - \frac{1}{2}) & +B \\ B & -\varepsilon T_0 (N_t - \frac{1}{2}) \end{pmatrix} \quad (3.7)$$

$$= a_{N_t}(t) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = a_{N_t}(t) \sigma_z \quad (3.8)$$

where

$$a_{N_t}(t) = t - \varepsilon T_0 \left(N_t + \frac{1}{2} \right). \quad (3.9)$$

since it must commute with σ_z at time zero. Therefore the equation for $\tilde{\psi}(t)$ takes the form

$$i \frac{\partial \tilde{\psi}}{\partial t} = \sqrt{\varepsilon} A(t) \tilde{\psi} = \sqrt{\varepsilon} a(t) \sigma_z \tilde{\psi}. \quad (3.10)$$

This equation can be easily solved, and one can see that it does not produce any transition. Hence, the entire relevant dynamics is determined by the piecewise constant (in time) averaged Hamiltonian.

We turn now to the calculation of $V(t)$ of (3.7)

For this purpose we use the relation $T_0 \varepsilon^2 = 1$ and write

$$f_n = e^{-i[\varepsilon T_0^2 (n + \frac{1}{2}) \sigma_z + B T_0 \sigma_x]} = e^{i \left[\sigma_z + \frac{\sigma_x}{\varepsilon T_0 (n + \frac{1}{2})} \right] (n + \frac{1}{2})} \quad (3.11)$$

$$\text{we denote } V(t = n T_0) = V_n \quad (3.12)$$

$$V_{n+1} = f_n V_n \quad (3.13)$$

A very useful formula is [4] , p. 203

$$f(a + \vec{b} \cdot \vec{\sigma}) = \alpha + \vec{\beta} \cdot \vec{\sigma} \quad (3.14)$$

where

$$\alpha = \frac{1}{2} [f(a + b) + f(a - b)] \quad (3.15)$$

$$\vec{\beta} = \frac{\vec{b}}{2b} [f(a + b) - f(a - b)]. \quad (3.16)$$

In our case $f(x) = e^{-ix}$ and

$$a_n = 0 \quad \text{and} \quad \tilde{b}_n = \left(BT_0, 0, \varepsilon T_0^2 \left(n + \frac{1}{2} \right) \right) \quad (3.17)$$

$$b_n = T_0 \sqrt{B^2 + \varepsilon^2 T_0^2 \left(n + \frac{1}{2} \right)^2} \quad (3.18)$$

$$\alpha_n = \cos b_n \quad \vec{\beta}_n = -i \left(BT_0, 0, \varepsilon T_0 \left(n + \frac{1}{2} \right) \right) \frac{\sin b_n}{b_n} \quad (3.19)$$

$$= \frac{-i \sin b_n}{T_0} \left(\frac{BT_0}{\sqrt{(B)^2 + \varepsilon^2 T_0^2 \left(n + \frac{1}{2} \right)^2}}, 0, \frac{\varepsilon T_0^2 \left(n + \frac{1}{2} \right)}{\sqrt{B^2 + \varepsilon^2 T_0^2 \left(n + \frac{1}{2} \right)^2}} \right) \quad (3.20)$$

$$(3.21)$$

$$(3.22)$$

$$f_n = \cos b_n + \frac{-i \sin b_n}{b_n} \left(BT_0 \cdot \sigma_x + \varepsilon T_0^2 \left(n + \frac{1}{2} \right) \sigma_z \right) \quad (3.23)$$

Let us consider three different domains

$$\text{I} \quad T_0 \varepsilon n \gg B \quad (3.24)$$

$$\text{II} \quad T_0 \varepsilon n \sim B \quad (3.25)$$

$$\text{III} \quad T_0 \varepsilon n \ll B \quad (3.26)$$

First we obtain the expression for $f_{n, f_{n+1}}$, where f_n is given by 3.4 with
We begin by recalling the definitions:

$$T_0 \equiv \frac{1}{\sqrt{\varepsilon}} \quad (3.27)$$

$$b_n = \varepsilon T_0^2 \left(n + \frac{1}{2} \right) \sqrt{1 + \frac{B^2}{\varepsilon^2 T_0^2 \left(n + \frac{1}{2} \right)^2}} \quad (3.28)$$

In domain I

$$\frac{\varepsilon T_0 \left(n + \frac{1}{2} \right)}{B} \gg 1 \quad (3.29)$$

Therefore

$$b_n = \varepsilon T_0^2 + \frac{B^2}{2\varepsilon \left(n + \frac{1}{2} \right)} + O \left(\frac{B^2}{\varepsilon^2 T_0^2 n} \right)^2 n \quad (3.30)$$

For future use, we write f_n in the following forms

$$f_n \equiv e^{-i(\varepsilon T_0^2(n+\frac{1}{2})\sigma_z + BT_0\sigma_x)} \quad (3.31)$$

$$= e^{-iT_0(\sqrt{\varepsilon}(n+\frac{1}{2})\sigma_z + B\sigma_x)} \quad (3.32)$$

$$= e^{-iT_0\sqrt{\varepsilon}(n+\frac{1}{2})\left[\sigma_z + \frac{B}{\sqrt{\varepsilon}(n+\frac{1}{2})}\sigma_x\right]} \quad (3.33)$$

$$f_n \equiv e^{-i\sqrt{\varepsilon}T_0(n+\frac{1}{2})[\sigma_z + \varepsilon_n\sigma_x]} \quad (3.34)$$

$$\text{with } \varepsilon_n = \frac{B}{\sqrt{\varepsilon}(n+\frac{1}{2})} \quad (3.35)$$

$$(3.36)$$

f_n of 3.23 can be written as

$$f_n = \left[\cos b_n - \frac{i}{b_n} \sin b_n \varepsilon T_0^2 \left(n + \frac{1}{2} \sigma_z \right) \right] \times \left[1 - \left[\cos b_n - \frac{i}{b_n} \sin b_n T_0^2 \left(n + \frac{1}{2} \right) \sigma_z \right]^{-1} \frac{i}{b_n} \sin b_n B T_0 \sigma_x \right] \quad (3.37)$$

where b_n is given by (3.18). We define now b'_n by

$$e^{-ib'_n\sigma_z} \equiv \left[\cos b_n - \frac{i}{b_n} \sin b_n \varepsilon T_0^2 \left(n + \frac{1}{2} \right) \sigma_z \right] \quad (3.38)$$

then

$$f_n = \bar{e}^{ib'_n\sigma_z} \left[I + e^{ib'_n\sigma_z} i \frac{B T_0}{b_n} \sin b_n \sigma_x \right] \quad (3.39)$$

where its main part is given by

$$F(b'_n) = e^{-ib'_n\sigma_z}$$

and ε_n is given by 3.35. note that b'_n is defined through f_n so that approximately for large n $e^{-ib_n\sigma_z}$ or $b'_n \sim b_n$. The reason it can be done is that b_n grows as n . We introduce the small correction.

$$e^{+ib'_n\sigma_z} \frac{B T_0}{b_n} \sin b_n \sigma_x \equiv \varepsilon_n^X. \quad (3.40)$$

We define $t_n = T_0\sqrt{\varepsilon}(n+\frac{1}{2})$ and turn to the calculation of the product, using Lemma 2.2 (Eq ??,2.25) one finds

$$f_n f_{n+1} = e^{-it_n(\sigma_z + \varepsilon_n\sigma_x)} e^{-it_{n+1}(\sigma_z + \varepsilon_{n+1}\sigma_x)} \quad (3.41)$$

$$= e^{-it_n(\sigma_z + \varepsilon_n\sigma_x)} e^{-it_n\left(\frac{t_{n+1}}{t_n}\sigma_z + \frac{t_{n+1}}{t_n}\varepsilon_{n+1}\sigma_x\right)} \quad (3.42)$$

$$= \exp i \int_0^{t_n} C_n(s) ds \quad (3.43)$$

where

$$C_n(s) = \sigma_z + \varepsilon_n \sigma_x + e^{-is(\sigma_z + \varepsilon_n \sigma_x)} \left[\frac{t_{n+1}}{t_n} \sigma_z + \frac{t_{n+1}}{t_n} \varepsilon_{n+1} \sigma_x \right] e^{+is(\sigma_z + \varepsilon_n \sigma_x)} \quad (3.44)$$

Using 3.20 - 3.23 one finds

$$e^{-is(\sigma_z + \varepsilon_n \sigma_x)} = \frac{1}{2} e^{-is\sqrt{1+\varepsilon_n^2}} + \frac{1}{2} e^{is\sqrt{1+\varepsilon_n^2}} \quad (3.45)$$

$$- \left(i \sin s \sqrt{1+\varepsilon_n^2} \right) \frac{1}{\sqrt{1+\varepsilon_n^2}} (\sigma_z + \varepsilon_n \sigma_x) \quad (3.46)$$

$$= \cos s \sqrt{1+\varepsilon_n^2} - \frac{i}{\sqrt{1+\varepsilon_n^2}} \sin \sqrt{1+\varepsilon_n^2} s (\sigma_z + \varepsilon_n \sigma_x) \quad (3.47)$$

We turn to define ε'_n and $\tilde{\varepsilon}_n$

$$e^{-is\sqrt{1+\varepsilon_n^2}\sigma_z} \left(I + e^{is\sqrt{1+\varepsilon_n^2}\sigma_z} \frac{-i}{\sqrt{1+\varepsilon_n^2}} \sin \sqrt{1+\varepsilon_n^2} \varepsilon_n \sigma_x \right) \quad (3.48)$$

$$\equiv e^{-is\sqrt{1+\varepsilon_n'^2}\sigma_z} (I + \tilde{\varepsilon}_n(s) \sigma_x) \quad (3.49)$$

and it satisfies

$$\varepsilon'_n \approx \varepsilon_n$$

We now turn to calculate the leading term with the help of (3.48)

$$e^{-is(\sigma_z + \varepsilon_n \sigma_x)} \frac{t_{n+1}}{t_n} \sigma_z e^{is(\sigma_z + \varepsilon_n \sigma_x)} \quad (3.50)$$

$$= e^{-is\sqrt{1+\varepsilon_n'^2}\sigma_z} (I + \tilde{\varepsilon}_n \sigma_x) \left(\frac{t_{n+1}}{t_n} \right) \sigma_z (I - \tilde{\varepsilon}_n \sigma_x) e^{is\sqrt{1+\varepsilon_n^2}\sigma_z}. \quad (3.51)$$

Now (3.41) takes takes form

$$f_n f_{n+1} = \exp \left[-i \int_0^{t_n} C_n(s) ds \right] \quad (3.52)$$

with

(3.53)

$$C_n(s) = \sigma_z + \varepsilon_n \sigma_x + \frac{t_{n+1}}{t_n} \sigma_z + \frac{t_{n+1}}{t_n} F \left(s \sqrt{1 + \varepsilon_n'^2} \right) \varepsilon_{n+1} \sigma_x F^* \left(s \sqrt{1 + \varepsilon_n'^2} \right) + \quad (3.54)$$

$$F \left(s \sqrt{1 + \varepsilon_n'^2} \right) \tilde{\varepsilon}_n(s) \frac{t_{n+1}}{t_n} (-2i\sigma_y) F^* \left(s \sqrt{1 + \varepsilon_n'^2} \right) + O(\tilde{\varepsilon}_n^2(s)) \sigma_z \quad (3.55)$$

$$= \left[1 + \frac{t_{n+1}}{t_n} + O(\tilde{\varepsilon}_n^2) \right] \sigma_z + \varepsilon_n \sigma_x + i F \left(s \sqrt{1 + \varepsilon_n'^2} \right) \tilde{\varepsilon}_n \sigma_y F^* \left(s \sqrt{1 + \varepsilon_n'^2} \right) \left(\frac{-t_{n+1}}{t_n} \right) \quad (3.56)$$

$$+ F \left(s \sqrt{1 + \varepsilon_n'^2} \right) \tilde{\varepsilon}_{n+1} \sigma_x \left(\frac{+t_{n+1}}{t_n} \right) F^* \left(s \sqrt{1 + \varepsilon_n'^2} \right) \quad (3.57)$$

so,

$$f_n f_{n+1} = \mathcal{T} \exp \left[(-i)[t_n + t_{n+1} + O_n(\varepsilon)] \sigma_z + i \frac{t_n \varepsilon_n}{t_n + t_{n+1}} \sigma_x (t_n + t_{n+1}) \right] \quad (3.58)$$

$$\cdot (t_n + t_{n+1} + O(\varepsilon)) \frac{t_{n+1}}{t_n (t_n + t_{n+1} + O_n)} \quad (3.59)$$

$$\int_0^{t_n} F \varepsilon_{n+1} \sigma_x F^* ds + 2i \frac{t_{n+1}}{t_n} \int_0^{t_n} F \left(s \sqrt{1 + \varepsilon_n'^2} \right) \sigma_y F^* \left(s \sqrt{1 + \varepsilon_n'^2} \right) ds \quad (3.60)$$

$$= \mathcal{T} \exp(-i \left\{ [t_n + t_{n+1} + O(\varepsilon_n)] \left[\sigma_z + \frac{\varepsilon_n t_n}{O(\varepsilon_n) + t_n + t_{n+1}} \sigma_x + \frac{t_{n+1}}{O(\varepsilon_n) + t_n + t_{n+1}} \frac{1}{t_n} \int_0^{t_n} G(s) \dots \right] \right\}) \quad (3.61)$$

where

$$G(s) = \dots = \frac{1}{t_n} \int_0^{t_n} 2i F \varepsilon_n(s) \sigma_y F^* ds + \frac{1}{t_n} \int_0^{t_n} F \varepsilon_{n+1} \sigma_x F^* ds. \quad (3.62)$$

The expansion has two parts, the integral one which is time dependent part denoted by $G(s)$, the rest. Therefore the following Lemme is useful

Lemma 3.1.

Let $U(t) = \mathcal{T} e^{-iAt - i \int_0^t B(s) ds}$ with $B(t) = U(t) B U^*(t)$. Then $\frac{dU}{dt} = (-iA - i\beta(t))U$.

If $U = e^{-iAt} W(t)$ then

$$\dot{W} = -ie^{iAt} B(t) e^{-iAt} W \quad (3.63)$$

In our case

$$A = \sigma_z \left(1 + \frac{t_{n+1}}{t_n} \right) a(O(\epsilon_n) + O(\tilde{\epsilon}_n)) , \quad B(t) = C_n F \left(t \sqrt{1 + \epsilon_n'^2} \right) \tilde{\epsilon}_n(s) \sigma_y F^* \quad (3.64)$$

$$C_n = \frac{-t_{n+1}}{t_n} \cdot \frac{1}{t_n} \quad (3.65)$$

$$\Rightarrow \dot{W} \approx i C_n \tilde{\epsilon}_n(t) \sigma_y W. \quad (3.66)$$

Where we used that $e^{-At} F \approx 1$, and it is understood that $t = t_n$ is taken. Integrating, we get

$$W(t_n) = e^{-i C_n \sigma_y \epsilon_n \int_0^{t_n} \frac{1}{\sqrt{1 + \epsilon_n'^2}} \sin \sqrt{1 + \epsilon_n'^2} s ds} \quad (3.67)$$

$$= e^{-i \frac{t_{n+1}}{t_n} \sigma_y (\cos \sqrt{1 + \epsilon_n'^2} t_n - 1) \frac{\epsilon_n}{1 + \epsilon_n'^2} \frac{1}{t_n}}. \quad (3.68)$$

The integral is bounded therefore its contribution will turn out to be small. We arrived at

$$f_n f_{n+1} \sim \exp i \int_0^{t_n + t_{n+1} + O_n(\epsilon)} \tilde{D}_n ds$$

with

$$\tilde{D}_n \sim \sigma_z + \left(\frac{t_n}{t_n + t_{n+1} + O(\epsilon_n)} \epsilon_n \sigma_x + \frac{t_{n+1}}{t_n + t_{n+1} + O_n(\epsilon)} \epsilon_{n+1} \sigma_x \right). \quad (3.69)$$

What we did actually is extended the domain of integration from t_n to $t_n + t_{n+1}$ prefactor of the leading term was set to remain the same while the corrections are different. Hence to the leading order

$$f_n f_{n+1} \approx e^{-i(t_n + t_{n+1}) \sigma_z} \quad (3.70)$$

Iterating the process \tilde{D}_n is replaced by \tilde{D}_{n+1} so that The correction term decreases compared to the σ_z term. Continuing the process results in

$$f_n f_{n+1} f_{n+2} = \exp i \int_0^{t_n + t_{n+1} + t_{n+2} + O_{n+1}(\epsilon)} \tilde{D}_{n+1}(s) ds \quad \tilde{D}_{n+1}(s) \sim \quad (3.71)$$

$$\sigma_z + \sigma_x \left(\frac{t_n}{t_n + t_{n+1} + t_{n+2} + O_{n+1}(\epsilon)} \epsilon_n + \frac{t_{n+1}}{(\dots)} \epsilon_{n+1} + \frac{t_{n+2}}{(\dots)} \epsilon_{n+2} \right). \quad (3.72)$$

If we iterate the process starting from 3.41, to leading order one finds,

$$f_{N_0} f_2 f_3 \dots f_N \approx e^{-i(t_{N_0} \dots t_N) \sigma_z} + C.C \quad (3.73)$$

where N_0 is the minimal n in Domain I.

The first correction in 3.72 is of order $\frac{1}{N^2}$ since t_n is of order N_0 while ϵ_n is of order $\frac{1}{N_0}$. Therefore in the limit $N \rightarrow \infty$ it vanishes

The ϵ_n contributions decay with iterations, since each one is proportional to one time t_n , out of overall time $t_1 + t_2 + \dots + t_N$.

Therefore $V_n \approx e^{-iN\sigma_z}$ commuting with σ_z , based on (2.10),(2.18) we find

$$i\frac{\partial\tilde{\psi}}{\partial t} = \sqrt{\epsilon}A(t)\tilde{\psi} = \sqrt{\epsilon}a(t)\sigma_z\tilde{\psi}. \quad (3.74)$$

Domain III

In this Domain f_n takes the form

$$f_n = e^{-iBT_0\sigma_x} (I + \text{corrections}) \quad (3.75)$$

and this form holds for $n + \frac{1}{2} \ll BT_0$ where $T_0^2\epsilon = 1$ was used.

A procedure similar to the one that was used in domain I leads to

$$V_N \approx e^{-iNBT_0^2\sigma_x} \quad (3.76)$$

In this case A of (2.18) is

$$A = \sqrt{\epsilon}a_n(t) e^{iNBT_0\sigma_x}\sigma_z e^{-iNBT_0\sigma_x} \quad (3.77)$$

where a_n is given by (3.9).

In this Domain therefore, to the leading order ψ develops with a Hamiltonian proportional to σ_x while $\tilde{\psi}$ evolves by the Hamiltonian (3.77)

Domain II

In this domain there is no simple expression for V of 2.13b,2.13a and it reduces to products of matrices

4 Adiabatic Theorem

In this section, we give a new proof of the adiabatic theorem in the gapless case. Furthermore, we prove an ergodic theorem that holds uniformly for the dynamics generated by adiabatic Hamiltonians. In fact, the ergodic theorem implies the adiabatic theorem. We start with the formulation of the theorem.

Theorem 2.

The Adiabatic Theorem

Let $\{H(s) \mid s \in [0, 1]\}$ be a family of bounded self-adjoint operators on $\{\mathcal{H}\}$ - a separable Hilbert space.

Assume that

1. i) $H(s)$, $\dot{H}(s)$, $\ddot{H}(s)$ are bounded and continuous uniformly in $s \in [0, 1]$ (where $\dot{f} \equiv \frac{df}{dt}$).
2. ii) $H(s)$ has an eigenvector $\psi_0(s)$, with projection $P_0(s)$ on the bound state we focus on. It will be denoted ψ_0 here and in what follows.
 - (a) $P(s)$, $\dot{P}(s)$, $\ddot{P}(s)$ are bounded and continuous, uniformly in $s \in [0, 1]$.
 - (b) There is a minimal distance between the eigenvalue $\lambda_0(s)$ of $\psi_0(s)$ and any other eigenvalue of $H(s)$. $\lambda_0(s)$ may be embedded in the continuous spectrum of $H(s)$.

Then, if the initial condition of the Schrödinger equation ($\varepsilon > 0$, small)

$$i \frac{\partial \psi_\varepsilon}{\partial t} = H(\varepsilon t) \psi_\varepsilon \quad (4.1)$$

is $\psi(0) = \psi_0(t)$ at $t = 0$, we have that for all $0 < t \leq \frac{1}{\varepsilon}$.

$$\|\psi_\varepsilon(t) - e^{i\theta(t)} \psi_0(\varepsilon t)\| = o_\varepsilon(1) . \quad (4.2)$$

Here, $o_\varepsilon(1)$ stands for a function that vanishes as $\varepsilon \rightarrow 0$ and $\theta(t)$ is a real valued function of t

Proof

The proof follows the strategy of Kato[24], which reduces the problem to estimating the size of an appropriate wave operator. Let $U_K(t)$ stand for the associated Kato dynamics

$$i \frac{dU_k(t)}{dt} = K(t) U_k(t) \quad (4.3)$$

$$K(t) \equiv H(\varepsilon t) + i\varepsilon \left[\dot{P}_0(\varepsilon t), P_0(\varepsilon t) \right] \quad (4.4)$$

where $\dot{P}_0(\varepsilon t)$ stands for $\frac{\partial P_0(\mu)}{\partial \mu}|_{\mu=\varepsilon t}$ and $K(t)$ is the Hamiltonian of the Kato dynamics.

The main property of the Kato dynamics is that it evolves $P_0(0)$ to $P_0(\varepsilon t)$, unitarily:

$$P_0(\varepsilon t) U_k(t) = U_k P_0(0) \quad (4.5)$$

Then, the proof of the theorem follows from showing that

$$\|U^*(t) U_K(t) P_0(0) - P_0(0)\| \leq o_\varepsilon(1) \text{ for } 0 \leq t \leq 1/\varepsilon. \quad (4.6)$$

Writing the above as the integral of the derivative (Cook's method), the problem is reduced to proving that (see Eq. 4.17)

$$\|\varepsilon \int_0^{1/\varepsilon} U^*(t) \left[\dot{P}_0(\varepsilon t), P_0(\varepsilon t) \right] U_K(t) P_0(0) dt\| = o_\varepsilon(1). \quad (4.7)$$

A key observation for the proof is the following proposition, a consequence of the formula for integration by parts:

Proposition 1.

Let $A(t)$, $B(t)$ be families of bounded smooth operators and $0 < \varepsilon \ll 1$; then,

$$\begin{aligned} \varepsilon \int_0^{1/\varepsilon} A(t) B(t) dt &= \varepsilon \left(\int_0^s A(s') ds' \right) B(s) \Big|_{s=1/\varepsilon} \\ &\quad - \varepsilon \int_0^{1/\varepsilon} \left(\int_0^s A(s') ds' \right) \frac{dB(s)}{ds} ds \end{aligned} \quad (4.8)$$

Furthermore, assume that

$$\|\varepsilon \int_0^{1/\varepsilon} A(s) ds\| = o_\varepsilon(1) \quad (4.9)$$

and

$$\left\| \frac{dB(s)}{ds} \right\| = O(\varepsilon), \quad (4.10)$$

then

$$\|\varepsilon \int_0^{1/\varepsilon} A(t) B(t) dt\| = o_\varepsilon(1). \quad (4.11)$$

Proof. The proof starts with integration by parts.

To prove 4.11, we write the second term on the RHS of Eq. (4.8) as

$$\varepsilon \int_0^{1/\varepsilon} \left(\int_0^{1/\sqrt{\varepsilon}} A(s') ds' \right) \frac{dB(s)}{ds} ds + \varepsilon \int_0^{1/\varepsilon} \left(\int_{1/\sqrt{\varepsilon}}^s A(s') ds' \right) \frac{dB(s)}{ds} ds. \quad (4.12)$$

The norm of the first term in Eq (4.12) is bounded by

$$\sqrt{\varepsilon} \int_0^{1/\varepsilon} \left\| \sqrt{\varepsilon} \int_0^{1/\sqrt{\varepsilon}} A(s') ds' \right\| \left\| \frac{dB(s)}{ds} \right\| ds \leq \sqrt{\varepsilon} \int_0^{1/\varepsilon} O(\varepsilon) o_{\sqrt{\varepsilon}}(1) ds \leq \sqrt{\varepsilon} o_{\sqrt{\varepsilon}}(1). \quad (4.13)$$

The second term in (4.13) is bounded by

$$\int_0^{1/\varepsilon} O(\varepsilon) ds \sup_{\frac{1}{\varepsilon} \geq s \geq 1/\sqrt{\varepsilon}} \left\| \varepsilon \int_{1/\sqrt{\varepsilon}}^s A(s') ds' \right\| \leq \quad (4.14)$$

$$\sup_{\frac{1}{\sqrt{\epsilon}} \leq s \leq \frac{1}{\epsilon}} O(\epsilon) \left[\left\| \epsilon \int_0^{1/\sqrt{\epsilon}} A(s') ds' \right\| + \left\| \epsilon \int_0^s A(s') ds' \right\| \right] \leq \epsilon \frac{1}{\sqrt{\epsilon}} + \sup_{\frac{1}{\sqrt{\epsilon}} < s < \frac{1}{\epsilon}} \frac{O(\epsilon)}{s} \left\| \int_0^s A(s') ds' \right\| \leq O(\epsilon) \sigma_\epsilon(1)$$

This completes proof of proposition 1

□

If we now write the integrand of Eq. (4.7) as (using the unitarity of U)

$$U^*(t) \left[\dot{P}_0(\varepsilon t), P_0(\varepsilon t) \right] P_0(\varepsilon t) U(t) U^*(t) U_K(t) P_0 \quad (4.15)$$

and notice that

$$i \frac{d}{dt} U^*(t) U_K(t) P_0(0) = U^*(t) \left[H(\varepsilon t) - H(\varepsilon t) + i\epsilon \left[\dot{P}_0, P_0 \right] \right] U_K(t) P_0(0) \quad (4.16)$$

$$= \epsilon U^*(t) \left[\dot{P}_0, P_0 \right] U_K(t) P_0(0) = O(\varepsilon), \quad (4.17)$$

then (4.6) follows from the above proposition if we prove that

$$\| \epsilon \int_0^{1/\varepsilon} U^*(t) \left[\dot{P}_0(\varepsilon t), P_0(\varepsilon t) \right] P_0(\varepsilon t) U(t) dt \| = o_\varepsilon(1). \quad (4.18)$$

Now, using that $P_0(s)^2 = P_0(s)$ it follows that

$$\dot{P}_0 P_0 + P_0 \dot{P}_0 = \dot{P}_0, \quad \text{so} \quad (4.19)$$

$$P_0 \dot{P}_0 P_0 + P_0 \dot{P}_0 = P_0 \dot{P}_0 \quad (4.20)$$

$$\text{or} \quad P_0 \dot{P}_0 P_0 = 0, \quad (\dot{P}_0 P_0 - P_0 \dot{P}_0) P_0 = \dot{P}_0 P_0 \quad (4.21)$$

using the fact that

$$\sum_j P_j + P_c = I$$

where $\{P_j\}$ are the bound states and P_c is the projection on the continuum combined with (4.21) we find that the integrand of (4.18) is

$$U^*(t) \left\{ \sum_{j \neq 0} P_j(\varepsilon t) \dot{P}_0(\varepsilon t) P_0(\varepsilon t) + P_c(H(\varepsilon t)) \dot{P}_0(\varepsilon t) P_0(\varepsilon t) \right\} U(t). \quad (4.22)$$

:

Lemma 3.

Let H_1, H_2 bounded self adjoint . operators, and such that $\|H_1 - H_2\| \leq \delta$. Suppose

$$\left\| \frac{1}{T} \int_0^T A_1(s) ds \right\| \equiv \left\| \frac{1}{T} \int_0^T e^{iH_1 t} A e^{-itH_1} dt \right\| = o_T(1). \quad (4.23)$$

(that is, $o_T(1) \rightarrow 0$ as $T \rightarrow \infty$), where $A_j = e^{iH_j t} A e^{-iH_j t}$ with $j = 1, 2$. Then

$$\left\| \frac{1}{T} \int_0^T A_1(s) ds - \frac{1}{T} \int_0^T A_2(s) ds \right\| \leq \delta o_T(1) \quad (4.24)$$

where $o_T(1)$ is as above.

Remark

The Lemma does not require that the ergodic bound (Eq. (4.23)) holds for H_2 . It will follow if $\delta \lesssim 1$, and with the *same* rate function $o_T(1)$.

Proof

The difference in (4.24) can be written as

$$\begin{aligned} & \left\| \frac{1}{T} \int_0^T \left[e^{itH_2} A (e^{-itH_1} - e^{-itH_2}) + (e^{itH_1} - e^{itH_2}) A e^{-itH_1} \right] dt \right\| \\ &= \left\| \frac{1}{T} \int_0^T [A_1(t) (I - \Omega_{12}(t)) + (I - \Omega_{21}(t)) A_2(t)] dt \right\| \end{aligned} \quad (4.25)$$

Where $\Omega_{ij} = e^{iH_j t} e^{-iH_i t}$. The result now follows by applying the proposition, using our assumption that

$$\left\| \frac{1}{T} \int_0^T A_1(t) dt \right\| = o_T(1),$$

with similar relations for A_2 and

$$\begin{aligned} \left\| \frac{d}{dt} \Omega_{ij}(t) \right\| &= \left\| \frac{d}{dt} e^{iH_j t} e^{-iH_i t} \right\| = \left\| e^{iH_i t} i(H_i - H_j) e^{-iH_j t} \right\| \\ &\leq \|H_i - H_j\| \leq \delta, \end{aligned} \quad (4.26)$$

for $i, j = 1, 2; \quad i \neq j$.

□

We now turn back to slowly changing Hamiltonians. We saw that in this case the solution $\psi(t)$ is given by (see Sec.2, 2.25 and 2.26)

$$\psi(t) = U(t)\psi_0 = V(t)\tilde{U}(t)\psi_0$$

with

$$U(t) = V(t) \cdot \tilde{U}(t)$$

The evolution of the averaged Hamiltonian is given by $V(t)$ while the correction is given by $\tilde{U}(t)$. The adiabaticity is reflected by mostly constant \tilde{U} and more precisely

$$\left\| \frac{d\tilde{U}(t)}{dt} \right\| \leq \text{const} \beta = \text{const} \sqrt{\epsilon}, \quad 0 \leq t \leq \frac{1}{\epsilon}$$

and so, now let us define the averaged Hamiltonian

$$\begin{aligned} \bar{A}(T) &\equiv \frac{1}{T} \int_0^T U^*(t) A U(t) dt \\ &= \frac{1}{T} \int_0^T \tilde{U}^*(t) V^*(t) A V(t) \tilde{U}(t) dt \\ &\equiv \frac{1}{T} \int_0^T \tilde{U}^*(t) A_V(t) \tilde{U}(t) dt. \end{aligned} \tag{4.27}$$

where

$$A_V(t) \equiv V^*(t) A V(t). \tag{4.28}$$

To estimate $\bar{A}(T)$, we break the sum into $N = \left\{ \frac{1}{\sqrt{\epsilon}} \right\}$ intervals of size $\frac{1}{\sqrt{\epsilon}}$ where $\{x\}$ is the closest integer to x .

We write:

$$\epsilon \int_0^{\frac{1}{\epsilon}} \tilde{U}^*(t) V^*(t) A V(t) \tilde{U}(t) dt \tag{4.29}$$

$$= \sqrt{\epsilon} \sum_{j=0}^{N-1} \sqrt{\epsilon} \int_{j \frac{1}{\sqrt{\epsilon}}}^{(j+1) \frac{1}{\sqrt{\epsilon}}} \tilde{U}^*(t) A_V(t) \tilde{U}(t) dt. \tag{4.30}$$

Therefore, if we can prove that

$$\sqrt{\epsilon} \int_{j \epsilon^{-\frac{1}{2}}}^{(j+1) \epsilon^{-\frac{1}{2}}} \tilde{U}^*(t) A_V(t) \tilde{U}(t) dt \leq o_{\sqrt{\epsilon}}(1), \tag{4.31}$$

uniformly in j , the ergodic theorem will follow for the pair $(U(t), A)$.

By the proposition 1 provided

$$\left\| \frac{d\tilde{U}(s)}{ds} \right\| \leq \sqrt{\epsilon} = \beta \tag{4.32}$$

and using the proposition 1 for \tilde{U} and \tilde{U}^* the result (4.31) follows from

$$\left\| \sqrt{\epsilon} \int_{j \frac{1}{\sqrt{\epsilon}}}^{(j+1) \frac{1}{\sqrt{\epsilon}}} A_V(s) ds \right\| = o_{\sqrt{\epsilon}}(1) \quad (\text{uniformly in } j) \tag{4.33}$$

Since condition (4.32) is satisfied by construction of the averaging, it remains to verify the estimate 4.33 on the piecewise constant dynamics $V(t)$.

Next we prove:

Theorem 4. *Uniform Ergodic Theorem*

Under our previous assumptions on the family of Hamiltonians $\{H(s)|0 \leq s \leq 1\}$, suppose A is a compact operator, and denote by $P_c(t)$ the projection on the Hilbert subspace of continuous spectrum of $\mathcal{H}_c(H(\varepsilon t))$. Then,

$$\left\| \frac{1}{T} \int_0^T U^*(t) A P_c(\varepsilon t) U(t) dt \right\| = o_T(1) \quad (4.34)$$

where $T = \frac{1}{\varepsilon}$, $U(t)$ is generated by the family of Hamiltonians $H(\varepsilon t)$, for all ε sufficiently small.

Proof

By the discussion above Eqs (4.30 - 4.32), we need only to show that

$$\left\| \sqrt{\varepsilon} \int_{j\varepsilon^{-\frac{1}{2}}}^{(j+1)\varepsilon^{-\frac{1}{2}}} V^*(t) A P_c(t) V(t) dt \right\| = o_\varepsilon(1) \text{ uniformly in } 0 \leq j \leq \varepsilon^{-\frac{1}{2}} \quad (4.35)$$

Now, recall that (Sec 2 and [1])

$$V(s) = V_n(s - nT_0) V_{n-1}(T_0) \cdots V_1(T_0) \quad (4.36)$$

where

$$V_i(s) = e^{-i\bar{H}_i(s)} \quad (4.37)$$

$$T_0 = \frac{1}{\sqrt{\beta}} = \varepsilon^{-\frac{1}{4}} \quad (\beta \equiv \sqrt{\varepsilon}). \quad (4.38)$$

\bar{H}_i is the averaged Hamiltonian in the time interval i . So, we have to evaluate

$$S_j = \sqrt{\varepsilon} \int_{jT_0^2}^{(j+1)T_0^2} V^*(s) A P_c(s) V(s) ds \quad (4.39)$$

$$\begin{aligned} &= \sqrt{\beta} \sum_{k=0}^{N-1} \sqrt{\beta} \int_{jT_0^2+kT_0}^{jT_0^2+(k+1)T_0} V_1^*(T_0) \cdots V_{n_{jk}}^*(s - n_{jk}T_0) A P_c(s) \cdot \\ &V_{n_{jk}} \cdots V_1(T_0) ds \end{aligned} \quad (4.40)$$

We divided the j -th interval into $N = \{T_0\}$ subintervals. In each of these the averaged Hamiltonian is constant.

Taking the norm of the above, we get the bound on each term in the sum over j is :

$$|S_j| \leq \frac{1}{T_0^2} T_0 \sup_j \left\| \int_{T_{jk}}^{T_{jk}+T_0} V_{n_{jk}}^* (s - n_{jk}T_0) AP_c(s) V_{n_{jk}} (s - n_{jk}T_0) ds \right\| \leq \quad (4.41)$$

$$\leq \sup_{\substack{j, k \\ k \geq 1}} o_{T_{jk}}(1), \quad (4.42)$$

n_j is the index of the j, k interval starting at time $T_{jk} = jT_0^2 + kT_0$. The last inequality results from the ergodic theorem for systems with time independent Hamiltonians [10]

It is left to show uniformity of the estimate in j .

Lemma \bar{H}

$$\bar{H}_j = \frac{1}{T_0^2} \int_{jT_0^2}^{(j+1)T_0^2} H(\varepsilon s) ds = H(y_j) + \frac{\sqrt{\varepsilon}}{2} H'(y_j) + \sup_y \|\ddot{H}(y)\| O_H(\varepsilon) \quad (4.43)$$

Where $O_H(\varepsilon)$ is a bound operator with norm $O(\varepsilon)$

Proof

$$\bar{H}_j = \frac{1}{T_0^2} \frac{1}{\varepsilon} \int_{y_j}^{y_j+\sqrt{\varepsilon}} H(y) dy = \frac{1}{\sqrt{\varepsilon}} \int_{y_j}^{y_j+\sqrt{\varepsilon}} H(y) dy \quad (4.44)$$

$$= \frac{1}{\sqrt{\varepsilon}} \left[\int_{y_j}^{y_j+\sqrt{\varepsilon}} H(y_j) dy + \int_{y_j}^{y_j+\sqrt{\varepsilon}} \dot{H}(y_j) (y - y_j) dy \right] \quad (4.45)$$

$$+ \int_{y_j}^{y_j+\sqrt{\varepsilon}} \frac{1}{2!} \ddot{H}(\tilde{y}) (y - y_j)^2 dy \quad (4.46)$$

where

$$y \equiv \varepsilon s, \quad y_j \equiv \varepsilon j T_0^2 = \varepsilon j \varepsilon^{-\frac{1}{2}} = \varepsilon^{\frac{1}{2}} j, \quad \tilde{y} = \tilde{y}(y) \subset [y_j, y_j + \sqrt{\varepsilon}]. \quad (4.47)$$

The statement of the Lemma then follows by our previous boundless assumption

$$\|H\| < C \quad (4.48)$$

$$\|\dot{H}\| < C \quad (4.49)$$

$$\|\ddot{H}\| < C \quad (4.50)$$

and

$$\sup_y |y - y_j| \leq \sqrt{\varepsilon}. \quad (4.51)$$

□

To continue the proof of theorem 4, assume $A = A(\varepsilon t)$, with $\sup_{0 \leq s \leq 1} \|\dot{A}(s)\| < C < \infty$.

To prove the uniformity in (j, k) of the estimate in (4.41), it is sufficient to prove the uniformity in $s \in [0, 1]$ of the bound on (4.53) with $T = 1/\varepsilon$.

Using the known ergodic (RAGE) estimate for time independent hamiltonians due to [9, 11] $[\epsilon_m, E - V_{ec}]$, the needed bound for each *fixed* s , will follow if the operator AP_c is compact and time independent.

To this end we write

$$A(\varepsilon t) P_c(\varepsilon t) = A(s) P_c(s) + \int_{\varepsilon^{-1}s}^t \frac{d}{dt''} A(\varepsilon t'') P_c(\varepsilon t'') dt'' \quad (4.52)$$

Using the assumed uniform boundedness of the derivatives \dot{A}, \dot{P}_c , the bound (4.54) - (4.55) follows.

Now using the ergodic theorem (RAGE) for (4.54), given any $\eta > 0$, it follows that for all $T \geq T_\eta(s)$, the term (4.54) is bounded by η . Hence (4.53) follows.

Given $\eta > 0$, then for all $s \in [0, 1]$, there exists time $T_\eta(s)$ s.t. $(t' = t - \varepsilon^{-1}s)$

$$\left\| \frac{1}{T_\eta(s)} \int_0^{T_\eta(s)} e^{iH(s)t'} A(\varepsilon t) P_c(\varepsilon t) e^{-iH(s)t'} dt' \right\| \quad (4.53)$$

$$\leq \left\| \frac{1}{T_\eta(s)} \int_0^{T_\eta(s)} e^{iH(s)t'} A(s) P_c(H(s)) e^{-iH(s)t'} dt' \right\| \quad (4.54)$$

$$+ \left\| \frac{1}{T_\eta(s)} \int_0^{T_\eta(s)} T_\eta(s) \varepsilon \left[\|A\| \|\dot{P}(s)\| + \|\dot{A}\| \|P(s)\| \right] dt' \right\| \quad (4.55)$$

$$\leq \eta + T_\eta(s) \varepsilon \sup_{s \in [0, 1]} \left[\|A\| \|\dot{P}(s)\| + \|\dot{A}(s) P_c(s)\| \right]. \quad (4.56)$$

The continuity of $H(s), A(s)$ in s , implies that if $|s - s'|$ is sufficiently small, then for all such s, s' , $\|H(s) - H(s')\| < \frac{1}{10T_\eta(s)}$ and $\|A(s) - A(s')\| \leq \frac{\eta}{2}$. By Lemma 3 the estimate (4.24) holds for all such $H(s')$, with the same $T_\eta(s)$. Note that the expression of (4.54) is equal to

$$\left\| \frac{1}{T_\eta(s)} \int_{s\varepsilon^{-1}}^{T_\eta(s) + \varepsilon^{-1}s} e^{iH(s)t'} AP_c(H(s)) e^{-iH(s)t'} dt' \right\| \quad (4.57)$$

since change of variables $t' \rightarrow t' - \varepsilon^{-1}s$ produces factors $e^{t'iH(s)(\varepsilon^{-1}s)}$ which drop due to unitarity.

Therefore, by the compactness of $[0, 1]$, it follows that there is a finite subcover of such intervals, centered at $\{s_i\}_{i=1}^K$, $K < \infty$.

Hence, for $T_{\max} = \sup_{s_i} T_{\frac{\eta}{2}}(s_i)$, we have that for all $T_{\frac{\eta}{2}} \geq T_{\max}$

$$\left\| \frac{1}{T_{\eta}} \int_{s\epsilon^{-1}}^{\epsilon^{-1}s+T_{\eta}} e^{+iH(s)t'} A(\epsilon t) P_c(t) e^{-iH(s)t'} dt \right\| < \frac{\eta}{2} + C\epsilon \frac{T_{\eta}}{2} \quad (4.58)$$

Choose $\epsilon^{-1} = T_{\max}^4$.

By Lemma \bar{H} , for any $y_i \leq 1/\epsilon$, $T_0 \equiv \epsilon^{-1/2}$ we have :

$$\|\bar{H}_j - H(y_j)\| \leq O(\sqrt{\epsilon}) \quad (4.59)$$

and therefore by Lemma 3 we can apply the above estimate (4.58) for all $y_j \leq \frac{1}{\epsilon}$ and $T_{\eta} \leq O(\epsilon^{-1/2}) = O(T_{\max}^2)$, choosing $s\epsilon^{-1} \equiv y_j$ in equation (4.47).

Collecting all of the above, we conclude that (choosing $T_{\frac{\eta}{2}} = \epsilon^{-1/2}$)

$$\left\| \sqrt{\epsilon} \int_{s\epsilon^{-1}}^{\epsilon^{-1}s+\epsilon^{-1/2}} e^{iH(s)t'} A(st) P_c(H(\epsilon t)) e^{-iH(s)t'} dt' \right\| < \eta$$

uniformly in s .

By (4.59) and Lemma 3, we can replace $H(s)$ by \bar{H}_j for $y_j = \epsilon^{1/2}j = \epsilon s$

Remark as can be seen from the proof, the compact operator A can be adiabatically time dependent.

Proof of the Gapless Adiabatic Theorem

To complete the proof of the theorem we need to show the estimate (4.6). By the properties of the Kato dynamics, the integrand can be written as (4.22).

We need to prove (4.7). The integrand of (4.7) is given by the expression 4.22. The second term in 4.22 is controlled by the above ergodic theorem,

since by assumption P_0 is finite rank and hence compact. Moreover,

$$\dot{P}_0(\epsilon t)P_0(\epsilon t) - \dot{P}_0(y_j)P_0(y_j) = O(\epsilon T_0) \text{ if } |\epsilon t - y_j| \leq T_0 \quad (4.60)$$

by arguments like (4.52)

$$\partial_t \left(\dot{P}(\epsilon t)P(\epsilon t) \right) = O(\epsilon) \left(\|\dot{P}\| + \|\ddot{P}\| \right). \quad (4.61)$$

To deal with the first term in the expression 4.22, the term

$$U^*(t) \sum_{j \neq 0} P_j(\epsilon t) \dot{P}_0(\epsilon t) P_0(\epsilon t) U_K(t) \psi_{E_0} \quad (4.62)$$

(we can put the $U_K(t)$ dynamics back again) we approximate $U^*(t)P_j$ by Kato's dynamics again:

$$U^*(t)P_j(\epsilon t) = U^*(t)U_{K_j}(t)U_{K_j}^*(t)P_j(\epsilon t) \quad (4.63)$$

with $U_{K_j}(t)$ generated by $K_j = \lambda_j(\epsilon t') + i\epsilon [\dot{P}_j(\epsilon t'), P_j(\epsilon t')]$ so that

$$U_{K_j}(t) = \mathcal{T} e^{-i \int_0^t \lambda_j(\epsilon t') - i\epsilon [\dot{P}_j(\epsilon t'), P_j(\epsilon t')] dt'} \quad (4.64)$$

Then, the expression (4.62) takes the form

$$e^{i \int_0^t \lambda_j(s) ds} [U(t)^* U_{K_j}(t)] e^{-i \int_0^t \lambda_j(s) ds} U_{K_j}^* P_j(\epsilon t) \dot{P}_0(\epsilon t) P_0(\epsilon t) U_{K_0}(t) \psi_{E_0} \quad (4.65)$$

We integrate this last expression by parts, where the first integration is of the composite integration of the phase factor. The rest of the terms are (integrals of) derivatives of

$$[U(t)^* U_{K_j}(t)], e^{-i \int_0^t \lambda_i(s) ds} U_{K_j}^*(t) P_j(\epsilon t) \dot{P}_0(\epsilon t) P_0(\epsilon t) U_{K_0}(t) \psi_0 \quad (4.66)$$

and are all of order ϵ (see 4.17 and 4.41). Without the loss of generality we assume $\lambda_0 = 0$. The composite integration gives

$$\int_0^t dt' e^{i \int_0^{t'} \lambda_j(\epsilon s) ds} = \frac{1}{(i\lambda_j(\epsilon t))} e^{i \int_0^t \lambda_j(\epsilon s) ds} \quad (4.67)$$

$$- \frac{1}{(i\lambda_j(0))} + \int_0^t \frac{dt'}{i\lambda_j^2} \epsilon \dot{\lambda}_j(\epsilon t') e^{i \int_0^{t'} \lambda_j(\epsilon s) ds}. \quad (4.68)$$

Since we assume that there is a uniform gap between the eigenvalues, $|\lambda_j| \geq \delta > 0$, and therefore the above expression is of order $\frac{1}{\delta^2}$.

We conclude that the contribution of (4.62) to (4.7) is of order ϵ/δ^2 as is the case for Adiabatic theorem with a gap.

□

Finally, we remark that if there is an eigenvalue crossing, finitely many times, the argument of Kato applies to give another correction of order $o_\epsilon(1)$.

5 Scattering Theory

Scattering theory provides an important theoretical and computational tool for analyzing adiabatic Hamiltonians. Recently, it became of crucial importance to understand these processes in the study of soliton (and other coherent) dynamics. Typically, linearization around soliton, and derivation of the modulation equation for its parameters lead to matrix valued adiabatic Hamiltonians. Our goal in this section is to show how the method of multi-time scale averaging, combined with the Uniform Ergodic Theorem for adiabatic Hamiltonians lead to scattering theory. In particular, we prove some classical propagation estimates and asymptotic completeness.

The analysis of time dependent potential scattering is complicated by the fact that energy is not conserved and cannot be localized. Some parts of the solution run away to infinity in the energy, while other parts concentrate on zero!

Propagation estimates were thus used, which included asymptotic localization of the momentum/energy. In some special cases, where the interaction is decaying fast in time enough one can prove the existence of asymptotic distribution of energies [19, 8, 9, 11, 23, 21, 22] . Here, we show how the Ergodic theorem provides another route.

Consider the case where there are no bound states. The general case will be considered elsewhere. $U(t)\psi$ has no bound states. Can we prove scattering for $U(t)\psi$, where $U(t)$ is generated by $H(\varepsilon t)$? We assume $H(\varepsilon t) = -\Delta + V(\varepsilon t, x)$, where

$$\sup_t \|\langle x \rangle^\sigma V(\varepsilon t, x)\|_{L^\infty} + \sup_t \left\| \langle x \rangle^\sigma \frac{\partial V(\tau, x)}{\partial \tau} \right\|_{L^\infty} < C_0. \quad (5.1)$$

First we notice, that a small localized perturbation does not change the scattering estimates in favorable cases.

Proposition 2. *Assume $\psi = U^{(0)}(t)\psi$ solves the equation*

$$i \frac{\partial \psi}{\partial t} = H_0(t)\psi_0, \quad \psi_0 = \psi(t=0) \quad (5.2)$$

and such that

$$\|\langle x \rangle^{-\sigma} \psi(t)\|_{L^2} \leq C_{\sigma,2} \langle t \rangle^{-\alpha}, \quad \alpha > 1. \quad (5.3)$$

Let $W_\varepsilon(x, t)$ be a localized function, bounded

$$\sup_t \|\langle x \rangle^{+2\sigma} W_\varepsilon(x, t)\|_{L^\infty} < C_0 \varepsilon. \quad (5.4)$$

Then, if ε is small enough, depending on $C_0, C_{\sigma,2}$, we have

$$\|\langle x \rangle^{-\sigma} U(t)\psi_0\|_{L^2} \leq C \langle t \rangle^{-\alpha}, \quad (5.5)$$

where $U(t)\psi_0$ solves the equation

$$i \frac{\partial}{\partial t} (U(t)\psi_0) = (H_0(t) + W_\varepsilon(x, t)) U(t)\psi_0. \quad (5.6)$$

Proof.

$$\psi(t) \equiv U(t)\psi_0 = U^{(0)}(t)\psi_0 - i \int_0^t U^{(0)}(t-s) W_\varepsilon(x, s) \psi(s) ds.$$

Then

$$\|\langle x \rangle^{-\sigma} \psi(t)\|_{L^2} \leq \|\langle x \rangle^{-\sigma} U^{(0)}\psi(t)\| + \int_0^t \|\langle x \rangle^{-\sigma} U^{(0)}(t-s) \langle x \rangle^{-\sigma_1}\|_{L^2} \quad (5.7)$$

$$\times \|\langle x \rangle^{+\sigma_1} W_\varepsilon(x, s) \langle x \rangle^{+\sigma_2}\|_{L^\infty} \|\langle x \rangle^{-\sigma} \psi(s)\|_{L^2} ds \quad (5.8)$$

$$\leq C_{\sigma_2} \langle t \rangle^{-\alpha} + C_{\sigma,2} \int_0^t \langle t-s \rangle^{-\alpha} \langle s \rangle^{-\alpha} C_{\sigma,2} \varepsilon \|\langle s \rangle^\alpha \langle x \rangle^{-\sigma} \psi(s)\|_{L^2} ds \quad (5.9)$$

Hence,

$$\sup_{t \leq T} \langle t \rangle^\alpha \left\| \langle x \rangle^{-\sigma} \psi(t) \right\|_{L^2} \leq C_{\sigma,2} + \epsilon C_0 C_{\sigma,2} \tilde{C} \sup_{0 \leq s \leq T} \left\| \langle s \rangle^\alpha \langle x \rangle^{-\sigma} \psi(s) \right\|_{L^2} \quad (5.10)$$

The result follows if $\epsilon C_0 C_{\sigma,2} \tilde{C} < 1$. \square

To Apply the above proposition to $U(t)\psi$ using the multiscale time averaging, we need to verify that

1. $U(t)\psi_0 = U_0(t)U_1(t) \dots U_N(t)\psi_0 = V_{0,\varepsilon}(t)\psi_0$ where the generator of $V_{0,\varepsilon}$ is the generator of $U_0(t)$ plus a small localized perturbation for all t .
2. That the scattering (local decay or similar) estimates hold for $U_0(t)\tilde{\psi}$.

We want to apply the Ergodic theorem to the dynamics above. We will assume $|V(x)| + |x \cdot \nabla V| \leq c \langle x \rangle^{-\sigma}$, $\sigma > 1$. The Ergodic theorem applies for any operator C s.t.

$$\sup_j \left\| \frac{1}{T} \int_0^T C e^{-iH_j t} dt \right\| = o_T(1), \quad (5.11)$$

so, it is sufficient to verify it for the time independent averaged Hamiltonians.

If C is compact, the result above is known as the RAGE theorem (see e.g., CFKS). We want to apply it to a noncompact operator C which is $O(\langle x \rangle^{-\sigma})$ for some $\sigma > 0$, *without* introducing high energy cutoff.

Local smoothing estimates hold for H_j : ($p \equiv -i\nabla_x$)

$$\int_{-\infty}^{\infty} \left\| \langle x \rangle^{-\frac{1}{2}-\varepsilon} |p|^{\frac{1}{2}} e^{-iH_j t} \psi \right\|_{L^2}^2 dt < C \|\psi\|_{L^2}^2. \quad (5.12)$$

For each T , $\frac{1}{T} \int_0^T e^{-iH_j t} \langle x \rangle^{-\sigma'} e^{iH_j t} dt$ is a bounded self-adjoint operator. Hence, for $\sigma > 1$,

$$\left\| \frac{1}{T} \int_0^T e^{iH_j t} \langle x \rangle^{-\sigma} e^{-iH_j t} dt \right\| \quad (5.13)$$

$$= \sup_{\|f\|_{L^2}=1} \frac{1}{T} \langle f, \int_0^T e^{iH_j t} \langle x \rangle^{-\sigma} e^{-iH_j t} f dt \rangle \quad (5.14)$$

$$= \sup_{\|f\|_{L^2}=1} \frac{1}{T} \int_0^T \left\| \langle x \rangle^{-\frac{\sigma}{2}} e^{-iH_j t} f \right\|^2 dt < O\left(\frac{1}{T}\right). \quad (5.15)$$

Therefore, in this case we have the following Ergodic estimate

$$\left\| \frac{1}{T} \int_0^T U^*(\varepsilon t) \langle x \rangle^{-\sigma} U(\varepsilon t) dt \right\| = O\left(\frac{1}{T}\right). \quad (5.16)$$

$$(5.17)$$

For a general Hamiltonian, restricted to the continuous spectrum , we can expect that $O\left(\frac{1}{T}\right)$ be replaced by

$$O\left(T^{-\alpha}\right), \quad \alpha \leq 1. \quad (5.18)$$

We have the following immediate result:

Theorem (Energy bound)

For the above system the “energy” can grow at most by $O(\varepsilon)$, up to time of order $\frac{1}{\varepsilon}$. Thus, the kinetic energy (the H^1 norm) can change by order 1.

Proof

$$\|U^*(\varepsilon t)H(\varepsilon t)U(\varepsilon t) - U^*(0)H(0)U(0)\| \quad (5.19)$$

$$= \left\| \int_0^t U^*(\varepsilon t) \frac{\partial H(\varepsilon t)}{\partial t} U(\varepsilon t) dt \right\| \quad (5.20)$$

$$= \left\| \int_0^t U^*(\varepsilon t) \frac{\partial W(x, \varepsilon t)}{\partial t} U(\varepsilon t) dt \right\| \quad (5.21)$$

$$\lesssim \varepsilon \left\| \int_0^t U^*(\varepsilon t) \langle x \rangle^{-\sigma} U(\varepsilon t) dt \right\| \leq O(\varepsilon). \quad (5.22)$$

Since $H(\varepsilon t) = -\Delta + V(x, \varepsilon t)$, with V bounded, the result follows. \square

Asymptotic Completeness

The proof of asymptotic completeness requires showing the following strong limit :

$$s - \lim e^{iH_0 t} U(t) \psi \quad (5.23)$$

exists for a dense set of ψ as $t \rightarrow \pm\infty$. By Cook’s method this is reduced to proving that

$$\int_0^\infty \|W(x, \varepsilon t)U(t)\psi\|^2 dt \leq c < \infty. \quad (5.24)$$

This last estimate follows directly from the above local decay estimate, provided the Hamiltonian becomes time independent after time $1/\varepsilon$.

Moreover, if the potential is slowly and adiabatically becoming time independent, that is if

$$W(x, \varepsilon t) \equiv W_0(x, \varepsilon t) + (1 + \varepsilon t)^{-a} W_1(x, \varepsilon t) \quad (5.25)$$

with W_0 time dependent up to time $1/\varepsilon$, and W_1 nonzero for all times; $a > 0$. This last statement follows by the following two simple observations: First , we can extend the proof of local decay estimate to arbitrary time of size M/ε , for M large, and fixed, by changing

ε to a smaller number, depending on M . Then, after time M/ε , the W_1 term is small, and absorbed as small perturbation of $H_0 = -\Delta + W_0(x, \infty)$ as shown in the beginning of the section.

This last example is already of considerable interest, since such Hamiltonians appear naturally in approximate nonlinear systems.

It is more difficult to get estimates for the case when the Hamiltonian does not turn off. In this case the standard Adiabatic Theorem does not apply, but one still would like to know the large time behavior of the system. This is considered below.

We will now use *monotonic propagation observables* [20]. The operator we use is of the form

$$\tanh(A/R). \quad (5.26)$$

Here A is the generator of dilation $A = \frac{1}{2}(x \cdot p + p \cdot x)$ and R is a (large) constant.

from this we derive the [20] following general estimate:

Theorem 5. *Let the dynamics $U(t)$ be generated by the Hamiltonian $-\Delta + W(x, t)$ and such that W is a sufficiently regular function, together with the derivatives $(x \cdot \nabla)^n W$, $0 \leq n \leq N, N > 3$. We also assume that W decays fast enough at infinity in x . Then, the following propagation estimate holds:*

$$\begin{aligned} \frac{1}{R} \int_0^T \left\langle p\psi(t), F\left(\frac{|A|}{R} \leq 1\right) p\psi(t) \right\rangle dt \leq \\ \frac{C}{R} \int_0^T \langle p\psi(t), < x >^{-\sigma} F(|x| \leq X_0) p\psi(t) \rangle dt + 2\|\psi\|^2, \end{aligned} \quad (5.27)$$

where σ is the decay rate of the potential, and for large enough (depending W) X_0

Theorem 6. *Assume $H(\epsilon t)$ is given as above and, $H(\epsilon t)$ is time dependent up to times $e^{\epsilon^{-\frac{1}{4}}}$*

Assume moreover that we have the following Ergodic type estimate:

$$\sup_n \int_{n/\sqrt{\epsilon}}^{(n+1)/\sqrt{\epsilon}} \langle p\psi(t), < x >^{-\sigma} p\psi(t) \rangle dt \leq c. \quad (5.28)$$

Then:

$$\int_1^T \left\langle p\psi(t), F\left(\frac{|A|}{R} \leq 1\right) p\psi(t) \right\rangle \frac{dt}{t} \leq c\|\psi\|^2.$$

Proof. Then Applying Theorem 5 with $(R \equiv T/\ln^2 T)$:

$$\frac{\ln^2 T}{T} \int_1^T \left\langle p\psi(t), F\left(\frac{|A|}{T/\ln^2 T} \leq 1\right) p\psi(t) \right\rangle dt \leq 2\|\psi\|^2 + c\sqrt{\epsilon} \frac{T \ln^2 T}{T}, \quad (5.29)$$

We therefore obtain, upon choosing $T \leq e^{1/\epsilon^{1/4}}$ that

$$\frac{\ln^2 T}{T} \int_1^T \left\langle p\psi(t), F\left(\frac{|A|}{R} \leq 1\right) p\psi(t) \right\rangle dt \leq c\|\psi\|^2, \quad (5.30)$$

for all

$$R \leq \frac{T}{\ln^2 T}.$$

Next, By integration by parts and the above estimate, we get

$$\int_1^T \left\langle p\psi(t), F\left(\frac{|A|}{R} \leq 1\right) p\psi(t) \right\rangle \frac{dt}{t} \leq c\|\psi\|^2. \quad (5.31)$$

Here R, T as above and this estimate is uniform in $\epsilon \downarrow 0$. \square

Theorem 7. (*Asymptotic completeness*)

AC holds for the above Hamiltonian (uniformly in $\epsilon \downarrow 0$) which depends on time up to $T \leq e^{\epsilon^{-1/4}}$.

Proof. We need to prove that for a dense set of ψ : $s - \lim U_0^*(t) U(t) \psi$ exists, uniformly in ϵ .

We break the expression above as

$$U_0^*(t) F\left(\frac{|A|}{R} \leq 1\right) U(t) \psi + U_0^*(t) F\left(\frac{|A|}{R} \geq 1\right) U(t) \psi \equiv I_1 + I_2$$

To prove that I_2 has a strong limit, we use cook's method:

$$\frac{d}{dt} U_0^*(t) F U(t) \psi = U_0^*(t) \{i[H_0, F] + FW\} U(t) \psi.$$

The first term $i[H_0, F] = \frac{1}{R} p F' \left(\frac{|A|}{R}\right) p + O(p^2/R^2)$. Therefore the first term is integrable by theorem 6.

The second term is of the form

$$\begin{aligned} & e^{iH_0 t} F\left(\frac{|A|}{R} > 1\right) \langle x \rangle^{-\sigma} U(t) \psi = \\ & e^{-iH_0 t} F\left(\frac{|A|}{R} > 1\right) F(|p| > kR) \langle x \rangle^{-\sigma} \frac{1}{|p|} |p| U(t) \psi \\ & + e^{iH_0 t} F\left(\frac{|A|}{R} > 1\right) F(|p| \leq kR) \langle x \rangle^{\sigma} U(t) \psi \equiv J_1 + J_2 \end{aligned}$$

The first term on the r.h.s is bounded by

$$\begin{aligned} \|J_1\|_{L^2} &\leq c \sup_{\|f\|} \frac{1}{kR} \left\| \langle x \rangle^{-\sigma/2} F e^{-iH_0 t} f \right\| \left\| \langle x \rangle^{-\sigma/2} (1 + |p|) U(t) \psi \right\| \\ \left(\int_0^T \langle f, J_1 \psi \rangle dt \right) &\leq \frac{1}{kR} \int_0^T \left\| \langle x \rangle^{-\sigma/2} F e^{-iH_0 t} f \right\|^2 dt + \frac{1}{kR} \int_0^T \left\| \langle x \rangle^{-\sigma/2} (1 + |p|) U(t) \psi \right\|^2 dt \end{aligned}$$

The first term is bounded for all T , a property of the free flow H_0 .

The second term, for $T \leq e^{1/\epsilon^{1/4}}$ is bounded by (using also 5.28)

$$\frac{c}{kR} \sqrt{\epsilon} T.$$

we used that $F(|p| > kR) \langle x \rangle^{-\sigma/2} |p|^{-1} = F|p|^{-1} \langle x \rangle^{-\sigma/2} + F \langle x \rangle^{-\sigma/2} \left[\langle x \rangle^{\sigma/2}, \frac{1}{|p|} \right] \langle x \rangle^{-\sigma/2} = O\left(\frac{1}{kR}\right) \langle x \rangle^{-\sigma/2}$. The second term J_2 is bounded by $(k \ll 1), \delta > 0$

$$\sim e^{iH_0 t} O(1) F(|x| > \delta R) \langle x \rangle^{-\sigma} U(t) \psi = O(\delta^{-\sigma} R^{-\sigma}). \quad (5.32)$$

By choosing the range of integration $T \leq e^{1/\epsilon^{1/4}}$ and $R = T/\ln^2 T$, $\sigma > 1$ we get that

$$J_1 \leq \frac{c}{k} \sqrt{\epsilon} \ln^2 T \leq \frac{c}{k}$$

$$J_2 \leq (\delta^{-\sigma} T^{-\sigma} \ln^{2\sigma} T) T \leq CT^{-a}, \quad a > 0.$$

Next we estimate I_1 .

We claim that $I_1 \rightarrow 0$ as $t \rightarrow \infty$; uniformly in $\epsilon \downarrow 0$.

Estimate (5.31) implies that

$$\frac{1}{T} \int_0^T \left\langle \psi(t), F(|p| \geq \delta) F\left(\frac{|A|}{R} \leq 1\right) F(|p| \geq \delta) \psi(t) \right\rangle dt \leq \frac{C}{\delta^2 \ln^2 T} \quad (5.33)$$

so, in particular $\left\langle \psi(t), F(|p| \geq \delta) F\left(\frac{|A|}{R} \leq 1\right) F(|p| \geq \delta) \psi(t) \right\rangle$ is small for t large.

The estimate I_1 :

$$I_1 \psi = e^{-iH_0 t} F\left(\frac{|A|}{R} \leq 1\right) F(p^2 \leq \epsilon) U(t) \psi$$

$$\|I_1 \psi\| \leq \|F(A \leq R) U(t) \psi\| = (U(t) \psi, F^2(A \leq R) U(t) \psi)^{1/2}$$

$$\begin{aligned} (U(t) \psi, F^2(A \leq R) U(t) \psi) - (U(0) \psi, F^2(A \leq R) U(0) \psi) &= \\ \int_0^t \langle \psi(t'), i[H(t'), F^2] \psi(t') \rangle dt' &= \int_0^t \left\langle \psi(t'), \frac{1}{R} p \tilde{F}(A) p \psi(t') \right\rangle dt' + \\ \int_0^t \langle \psi(t'), i[W(x, \epsilon t), F^2] A \rangle \psi(t') dt' & \end{aligned}$$

The first term is negative, as $\tilde{F} \sim F'^2(A)$. The second term the integrand is of the order $\langle \psi(t'), \frac{1}{R} \langle x \rangle^{-\sigma} F'(A) \psi(t') \rangle$. If we integrate the second term, the up to time $T = e^{\epsilon^{-1/4}}$ and use the assumption (5.28) with $R \sim T/\ln^2 T$, the integral of the W term is bounded by

$$C \frac{\ln^2 T}{T} \sqrt{\epsilon} T = C \cdot 1 = C < \infty$$

Since the first term is negative it is also uniformly bounded, by the uniform boundness of the R.H.S .

So the convergence is independent of ϵ .

Moreover if we let $t \rightarrow \infty$, then by assumption $H(\epsilon t) = H(\infty)$ for all $t > e^{\epsilon^{-1/4}}$.

Therefore, the integrand can be extended to $t = \infty$.

It follows that

$$U^*(t) F^2(A \leq R) U(t) \psi \xrightarrow{s} F^\pm \psi$$

as $t \rightarrow \pm\infty$

Moreover, since for all $|t| > e^{\epsilon^{-1/4}}$ the Hamiltonian is time independent and satisfies the usual scattering and decay estimates, it follows that $F^\pm \equiv 0$. \square

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